Lie-point symmetries and stochastic differential equations: II

2000 J. Phys. A: Math. Gen. 33 4883

(http://iopscience.iop.org/0305-4470/33/27/306)

J. Phys. A: Math. Gen. **33** (2000) 4883–4902. Printed in the UK PII: S0305-4470(00)12105-5

Lie-point symmetries and stochastic differential equations: II

Giuseppe Gaeta

Dipartimento di Fisica, Università di Roma I 'La Sapienza', P. le A Moro 5, I-00185 Roma, Italy

E-mail: gaeta@roma1.infn.it

Received 24 February 2000, in final form 24 May 2000

Abstract. We complement the discussion of symmetries of Ito equations given in Gaeta and Rodríguez Quintero (1999 *J. Phys. A: Math. Gen.* **32** 8485–505) by considering transformations acting on vector Wiener processes as well, together with discrete symmetries. We also discuss symmetries for the random dynamical system defined by an Ito equation, and show that there are, in general, more than the symmetries of the one-particle process defined by the same Ito equation.

Introduction

Symmetries of stochastic differential equations (Ito equations) in $\mathbb{R}^n$ have been defined and studied (and compared with symmetries of the associated Fokker–Planck equation) in a recent paper by Rodríguez Quintero and the present author [4]. We refer to that paper for motivation and a general qualitative discussion, just recalling that symmetry methods are a very valuable tool in the study of deterministic (ordinary or partial) differential equations [6].

It was pointed out in [4] that the class of symmetries considered there did not allow for transformations acting also on the n-dimensional vector Wiener process $\boldsymbol{w}(t)$ entering in the definition of the Ito equation; in particular, this prevented our method from recognizing the rotational invariance of some simple—and manifestly rotationally invariants—equations. Moreover, [4] was only concerned with symmetries of Ito equations considered as defining a one-particle process, and not seen as defining a random dynamical system [1].

In the present paper, we will complete the discussion of [4] by:

(a) enlarging the class of allowed transformations so as to include transformations which act on $\boldsymbol{w}(t)$ (section 1); and
(b) discuss symmetries of a random dynamical system defined by an Ito equation (section 2). For the sake of completeness, we also
(c) derive the determining equations for discrete symmetries of an Ito equation (in an appendix).

Our results will be illustrated by a number of examples on concrete Ito equations or classes of Ito equations (which are rather simple, but non-trivial and relevant in applications). These are given in section 3 for the symmetry of (one-particle processes defined by) Ito equations, and in section 4 for the symmetries of random dynamical systems (for which it will be sufficient to discuss two-particle processes). We will mostly resort to examples already analysed in [4], so that we can focus on 'new' symmetries without repeating computations for those already analysed there.

0305-4470/00/274883+20$30.00

As for the results contained in this paper, besides writing down the determining equations for the different classes of symmetries we consider, we show by means of examples that enlarging the class of admitted transformations so as to include those acting on the Wiener process $\boldsymbol{w}(t)$ (or, with a term we introduce below, admitting W-symmetries) does indeed in many cases lead to 'new' symmetries, not included in the classes studied in [4]. It should also be noted that this application of the class of admitted transformations does not change in any way the discussion given in [4] concerning the relations between the symmetries of an Ito equation (seen as defining a one-particle process) and of the associated Fokker–Planck equation (see remark 3 below). We also find, and again show by examples, that a random dynamical system defined by an Ito equation admits, in general, more symmetries than the one-particle process defined by the same equation.

It should be mentioned that this paper, similarly to [4], is only devoted to the determination and analysis of symmetries of an Ito equation (both in the sense of the one-particle process and of the random dynamical system it defines); that is, we do not discuss the way in which these symmetries could be used to extract meaningful information on the behaviour of the stochastic equation, similarly to what is done for deterministic differential equations [6], briefly mentioned in [4]. This matter will be discussed in a later contribution.

1. Symmetries involving the Wiener processes (W-symmetries)

We will consider symmetries involving not only the spatial and time variables $(\boldsymbol{x}, t)$, but also the vector Wiener processes $\boldsymbol{w}(t)$ entering in the n-dimensional Ito equation

$$\mathrm{d}x^i = f^i(\boldsymbol{x}, t)\,\mathrm{d}t + \sigma^i_{\ k}(\boldsymbol{x}, t)\,\mathrm{d}w^k(t). \tag{1.1}$$

In other words, we will consider infinitesimal transformations of the form

$$\begin{aligned} x^i &\to y^i = x^i + \varepsilon\xi^i(x, t) \\ t &\to s = t + \varepsilon\tau(t) \\ w^i &\to z^i = w^i + \varepsilon\mu^i(w, t). \end{aligned} \tag{1.2}$$

We also call symmetry generators of this form '*W-symmetries*'.

Note that this is not the most general possible form of a transformation for the variables involved in (1.1); a few words on such a restriction are in order.

The restriction on τ, i.e. the fact that the transformation of the time variable is not allowed to depend on the position $\boldsymbol{x}(t)$ nor on the realization of the Wiener process $\boldsymbol{w}(t)$ was already introduced in [4] and is entirely natural if we want to take into account the physical meaning of the time variable t: only time reparametrizations are allowed (see also appendix A of [4]).

In (1.2), we have also allowed the transformation of the vector Wiener process $\boldsymbol{w}(t)$ only as an 'internal' one; i.e. it cannot depend on the position $\boldsymbol{x}(t)$. The reason for this is analogous to that mentioned above: as we think of the stochastic process $\boldsymbol{w}(t)$ as independent of the evolution of the $\boldsymbol{x}(t)$, we want *a fortiori* its transformation not to depend on this.

Finally, we have allowed the transformation on the spatial coordinates $\boldsymbol{x}(t)$ to depend on $\boldsymbol{x}$ and t, but not on $\boldsymbol{w}$; this means that we do not want to consider transformations of the spatial coordinates which depend on the realization of the stochastic process $\boldsymbol{w}(t)$. This again is a natural requirement in physical terms, but it should be noted that in this way we are also discarding the transformations needed to obtain normal forms of stochastic differential equations (see [1, 2] and references therein).

Let us now compute how (1.1) changes under (1.2). We can first of all implement the change on $\boldsymbol{x}$ and t; repeating the computations given in [4] (see, in particular, section 3 therein) we have that (1.1) is changed into a new Ito equation for $\boldsymbol{y}(t)$, which we write as

$$\mathrm{d}y^i = \widetilde{f}^i(y,s)\,\mathrm{d}s + \widetilde{\sigma}^i_{\ k}(y,s)\,\mathrm{d}w^k \tag{1.3}$$

where the tilded functions are given by

$$\begin{aligned} \widetilde{f}^i(y,s) &= f^i(y,s) + \varepsilon(\delta f)^i \\ \widetilde{\sigma}^i_{\ k}(y,s) &= \sigma^i_{\ k}(y,s) + \varepsilon(\delta\sigma)^i_{\ k}. \end{aligned} \tag{1.4}$$

With the notation $A = \frac{1}{2}(\sigma\sigma^T)$, the variations are explicitly given by (see equation (3.4) in [4])

$$\begin{aligned} (\delta f)^i &= \partial_t\xi^i + [(f^j\cdot\partial_j)\xi^i - (\xi^j\cdot\partial_j)f^i] - \partial_t(f^i\tau) + A^{jk}\partial^2_{jk}\xi^i \\ (\delta\sigma)^i_{\ k} &= [(\sigma^j_{\ k}\cdot\partial_j)\xi^i - (\xi^j\cdot\partial_j)\sigma^i_{\ k}] - \tau\partial_t\sigma^i_{\ k} - \tfrac{1}{2}\sigma^i_{\ k}\partial_t\tau. \end{aligned} \tag{1.5}$$

Note that (1.3) is still expressed in terms of the 'old' Wiener processes $w^i(s)$; we have to express it in terms of the 'new' Wiener processes $z^i(s)$. We also stress that the effect of the change of the time variable ($t \to s$) in $\boldsymbol{w}$ has already been taken into account, conformly to the discussion in appendix A of [4], where it is shown that this amounts to a transformation of w^i into new processes $\widetilde{w}^i$, differing only by a scalar term $\mathrm{d}w^i = (1 - \varepsilon\partial_s\tau/2)\,\mathrm{d}\widetilde{w}^i$. The scalar term can thus be absorbed into the σ-matrix, as we have done in writing (1.5).

Let us now focus on the transformation $w^k \to z^k$; in (1.2) above we have $z^k = w^k + \varepsilon\mu^k(\boldsymbol{w},t)$, but actually $\mu(\boldsymbol{w},t)$ cannot be an arbitrary function. Indeed, $\boldsymbol{z}(t)$ must again be a vector Wiener process (in particular, all the momenta of $\boldsymbol{z}$ and $\mathrm{d}\boldsymbol{z}$ must coincide with those of $\boldsymbol{w}$ and $\mathrm{d}\boldsymbol{w}$); thus the only possibility is provided by constant orthogonal transformations,

$$z^k(t) = M^k_{\ j}w^j(t) \qquad MM^+ = I. \tag{1.6}$$

(The constancy in time of M is easily derived by considering $\langle z(t), z(t+\delta t)\rangle$.)

As we want to consider infinitesimal generators of such transformations, and to write the transformation from $\boldsymbol{w}$ to $\boldsymbol{z}$ in the form (1.2), this will be

$$z^k(t) = w^k(t) + \varepsilon B^k_p(t)w^p(t) \tag{1.7}$$

with $B(t)$ a real antisymmetric matrix. Therefore,

$$\mathrm{d}w^k = \mathrm{d}z^k - \varepsilon B^k_p\,\mathrm{d}z^p. \tag{1.8}$$

We can insert this into (1.3) and, using also (1.4) and (1.5) (i.e. the results of [4]), we obtain the complete expression for δf and $\delta\sigma$. This is summarized in the proposition below, where we also use the notation, already introduced in [4],

$$\{f,g\}^i := (f^j\cdot\partial_j)g^i - (g^j\cdot\partial_j)f^i. \tag{1.9}$$

Proposition 1. *Under the infinitesimal transformation $x^i \to y^i = x^i + \varepsilon\xi^i(\boldsymbol{x},t)$, $t \to s = t + \varepsilon\tau(t)$, $w^i \to z^i = w^i + \varepsilon B^i_k(t)w^k$, with B a real antisymmetric matrix, the Ito equation*

$$\mathrm{d}x^i = f^i(\boldsymbol{x},t)\,\mathrm{d}t + \sigma^i_k(\boldsymbol{x},t)\,\mathrm{d}w^k$$

is changed into a new Ito equation

$$\mathrm{d}y^i = F^i(\boldsymbol{y},s)\,\mathrm{d}s + S^i_k(\boldsymbol{y},s)\,\mathrm{d}z^k$$

with $F^i = f^i + \varepsilon(\delta f)^i$, *and* $S^i_k = \sigma^i_k + \varepsilon(\delta\sigma)^i_k$; *where the variations are given by*

$$\begin{aligned}(\delta f)^i &= \partial_s \xi^i + \{f,\xi\}^i - \partial_s(\tau f^i) + A^{jk}\partial^2_{jk}\xi^i \\ (\delta\sigma)^i_k &= \{\sigma_k,\xi\}^i - \tau\partial_s\sigma^i_k - \tfrac{1}{2}(\partial_s\tau)\sigma^i_k - \sigma^i_p B^p_k.\end{aligned} \tag{1.10}$$

Needless to say, equation (1.2) will be a symmetry of (1.1) only if it leaves this invariant, i.e. if $(\delta f)^i$ and $(\delta\sigma)^i_k$ vanish identically for all i and k [4].

Corollary. *The vector field* $X_0 = \xi^i(\boldsymbol{x},t)(\partial/\partial x^i) + \tau(t)(\partial/\partial t) + B^i_p(t)w^p(\partial/\partial w^i)$ *is a symmetry generator for the Ito equation (1.1) if and only if:*

(a) $B = -B^T$, *and*
(b) it satisfies the determining equations

$$\partial_t\xi^i + \{f,\xi\}^i - \partial_t(\tau f^i) + A^{jk}\partial^2_{jk}\xi^i = 0 \tag{1.11a}$$

$$\{\sigma_k,\xi\}^i - \tau\partial_t\sigma^i_k - \tfrac{1}{2}(\partial_t\tau)\sigma^i_k - \sigma^i_p B^p_k = 0. \tag{1.11b}$$

Remark 1. As B is an antisymmetric matrix, we are immediately guaranteed that it vanishes for one-dimensional equations (as is also obvious from its very definition). Thus for one-dimensional Ito equations we have no new symmetries with respect to those discussed in [4].

Remark 2. We have considered symmetries involving transformations of the vector Wiener process $\boldsymbol{w}$, but we actually proceeded in two steps: first considering 'old-type' transformations (involving only $\boldsymbol{x}$ and t), and then adding a transformation of the $\boldsymbol{w}$ as well. We stress that this was possible due to the specific form (1.2) of transformations we are allowing: had we considered $\xi = \xi(\boldsymbol{x},t;\boldsymbol{w})$ and/or $\mu = \mu(\boldsymbol{x},t;\boldsymbol{w})$ in (1.2), such a two-step procedure would not have been allowed. In particular, this would be the case for the transformations considered in taking an Ito equation into its normal form [1, 2].

Remark 3. If we isolate the effect of the 'second step' (see the previous remark) of our transformation on the Ito equation, this simply amounts to $\sigma \to \sigma M$, leaving f unchanged. As M is an orthogonal matrix, $A := \frac{1}{2}(\sigma\sigma^T)$ will not be affected by such a transformation. This means that if we just consider a $\boldsymbol{w}$ transformation $\boldsymbol{w} \to \boldsymbol{z} = M\boldsymbol{w}$ the Fokker–Planck equations associated with the Ito equation and to the transformed one will be the same. Thus all the discussion conducted in [4] about the relations between symmetries of an Ito equation and the associated Fokker–Planck equation, also applies to the present case; i.e. to transformations of the form (1.2), with absolutely no changes. Needless to say, for more general transformations (for which the two-steps procedure cannot be applied) this would not be true.

2. Symmetries of random dynamical systems

As pointed out—actually, thanks to a referee—in [4], the symmetries discussed there, and in the previous section, should be seen as symmetries of the *one-particle stochastic process* (OPP) defined by an Ito equation. The same equation also defines a *random dynamical system* (RDS). This can be thought of intuitively as the ensemble of stochastic processes undergone by indistinguishable particles with different initial conditions $\boldsymbol{x}(t_0)$ (corresponding to all points of $\mathbb{R}^n$) for the same realization of the n-dimensional vector Wiener process $\boldsymbol{w}(t)$. In more precise terms, an Ito equation also defines an N-point process in a realization of which N

particles with different initial position move simultaneously under the Ito equation with the *same* realization of $\boldsymbol{w}(t)$, with N arbitrary.

In this section we want to discuss symmetries of the RDS defined by an Ito equation, and how these relate to symmetries of the OPP defined by the same Ito equation. For ease of discussion, we will first focus on the two-point process defined by an Ito equation, and only later generalize the results to a general N-particle process (NPP).

Remark 4. For the wide (and relevant) class of RDS with independent increments, the full information contained in the RDS is actually embodied in the two-point process. We refer to [1] (in particular, section 2.3.9) for definitions and discussion.

Remark 5. The previous remark could dispense us from deriving the determining equations for NPP, but we will, however, prefer to give them explicitly. On the other hand, due to this remark we will use 'symmetry of the RDS' to mean the 'symmetry of the two-particle process'.

2.1. Symmetry of two-particle processes

We consider an Ito equation in $\mathbb{R}^n$,

$$\mathrm{d}x^i = \phi^i(\boldsymbol{x}, t)\,\mathrm{d}t + \rho^i_k(\boldsymbol{x}, t)\,\mathrm{d}w^k. \tag{2.1}$$

We want to consider the stochastic process undergone by two (non-interacting) particles evolving simultaneously—that is, for the same realization of $\boldsymbol{w}(t)$—under this equation.

We will denote the spatial coordinates of the first particle by $\boldsymbol{x} \in \mathbb{R}^n$, and those of the second one by $\boldsymbol{y} \in \mathbb{R}^n$. Needless to say, 'first' and 'second' are a completely arbitrary denomination, and all the equations (and results) should be covariant under the exchange $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$.

The corresponding evolution is described by a system of two n-dimensional Ito equations

$$\begin{aligned}\mathrm{d}x^i &= \phi^i(\boldsymbol{x}, t)\,\mathrm{d}t + \rho^i_k(\boldsymbol{x}, t)\,\mathrm{d}w^k\\ \mathrm{d}y^i &= \phi^i(\boldsymbol{y}, t)\,\mathrm{d}t + \rho^i_k(\boldsymbol{y}, t)\,\mathrm{d}w^k.\end{aligned} \tag{2.2}$$

Note that the Wiener processes $\boldsymbol{w}(t)$ appearing in the two equations are *the same*.

This system can be rewritten as a $(2n)$-dimensional Ito equation,

$$\mathrm{d}z^i = f^i(\boldsymbol{z}, t)\,\mathrm{d}t + \sigma^i_k(\boldsymbol{z}, t)\,\mathrm{d}v^k \tag{2.3}$$

where now $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n}$ and $\boldsymbol{v}$ is a $(2n)$-dimensional Wiener process, whose first n components are given by $\boldsymbol{w}$ (the other components will be inessential). For notational and typographical ease, we will now write x for $\boldsymbol{x}$, and so on.

We can rewrite the $2n$-dimensional vector f and the matrix σ appearing in (2.3) in (n-dimensional) block notation as

$$f = \begin{pmatrix}\phi(x, t)\\ \phi(y, t)\end{pmatrix} \qquad \sigma = \begin{pmatrix}\rho(x, t) & 0\\ \rho(y, t) & 0\end{pmatrix} \tag{2.4}$$

with the same notation, we also have that

$$\tfrac{1}{2}\sigma\sigma^T = \tfrac{1}{2}\begin{pmatrix}\rho(x, t)\,\rho^T(x, t) & \rho(x, t)\,\rho^T(y, t)\\ \rho(y, t)\,\rho^T(x, t) & \rho(y, t)\,\rho^T(y, t)\end{pmatrix} := \begin{pmatrix}A(x, x) & A(x, y)\\ A(y, x) & A(y, y)\end{pmatrix}. \tag{2.5}$$

We can now look for symmetries of (2.3) using the equations for symmetries of (OPP defined by) an Ito equation given in [4] or, more generally, those given in the previous section.

The requirement of covariance under the exchange of x and y implies that in doing this we can assume

$$\xi(x, y) = \begin{pmatrix} \eta(x, y, t) \\ \eta(y, x, t) \end{pmatrix}. \tag{2.6}$$

In the following we will keep track of x and y dependences but, for ease of notation and since all quantities depend on t, will drop the t dependences. We will also use n-vector notation throughout; ∇_x will be the gradient vector operator with respect to the x variables, and similarly for ∇_y. Similarly, $\{f, g\}_x$ will denote the bracket of f and g with respect to the x variables, $\{f, g\}_x := (f\nabla_x)g - (g\nabla_x)f$.

The term $\{f, \xi\}$ is easily computed by a straightforward substitution:

$$\begin{aligned}
\{f, \xi\} &= [\phi(x)\nabla_x + \phi(y)\nabla_y] \begin{pmatrix} \eta(x, y) \\ \eta(y, x) \end{pmatrix} - [\eta(x, y)\nabla_x + \eta(y, x)\nabla_y] \begin{pmatrix} \phi(x) \\ \phi(y) \end{pmatrix} \\
&= \begin{pmatrix} \phi(x)\nabla_x\eta(x, y) + \phi(y)\nabla_y\eta(x, y) - \eta(x, y)\nabla_x\phi(x) \\ \phi(x)\nabla_x\eta(y, x) + \phi(y)\nabla_y\eta(y, x) - \eta(y, x)\nabla_y\phi(y) \end{pmatrix} \\
&= \begin{pmatrix} \{\phi(x), \eta(x, y)\}_x + \phi(y)\nabla_y\eta(x, y) \\ \{\phi(y), \eta(y, x)\}_y + \phi(x)\nabla_x\eta(y, x) \end{pmatrix}.
\end{aligned} \tag{2.7}$$

We also have immediately

$$\partial_t\xi = \begin{pmatrix} \partial_t\eta(x, y) \\ \partial_t\eta(y, x) \end{pmatrix} \qquad \partial_t(\tau f) = \begin{pmatrix} \partial_t[\tau\phi(x)] \\ \partial_t[\tau\phi(y)] \end{pmatrix}. \tag{2.8}$$

As for the last term that we have to compute in order to obtain (1.11a), we will use a compact notation for the second derivatives operators: we write

$$A(x, y)\nabla^2_{xy} := \sum_{i=1}^{n}\sum_{j=1}^{n} A^{ij}(x, y)\frac{\partial^2}{\partial x^i\partial y^j}. \tag{2.9}$$

With this, we have

$$\begin{aligned}
&\sum_{j=1}^{2n}\sum_{k=1}^{2n} \tfrac{1}{2}(\sigma\sigma^T)^{jk}\partial^2_j k\xi \\
&= \begin{pmatrix} A(x, x)\nabla^2_{xx}\eta(x, y) + A(y, y)\nabla^2_{yy}\eta(x, y) + A(x, y)\nabla^2_{xy}\eta(x, y) + A(y, x)\nabla^2_{yx}\eta(x, y) \\ A(x, x)\nabla^2_{xx}\eta(y, x) + A(y, y)\nabla^2_{yy}\eta(y, x) + A(x, y)\nabla^2_{xy}\eta(y, x) + A(y, x)\nabla^2_{yx}\eta(y, x) \end{pmatrix}.
\end{aligned} \tag{2.10}$$

This completes the computations needed to write (the $2n$-dimensional vector) equation (1.11a). We can actually just write the first n-dimensional vector block: the second is obtained by this via the exchange of x and y, as can be checked by the previous computations. Putting together (2.7), (2.8) and (2.10), and reintroducing the t dependence for completeness, the determining equation is written as

$$\begin{aligned}
&\big[\{\phi(x, t), \eta(x, y, t)\}_x + \partial_t\eta(x, y, t) - \partial_t(\tau(t)\phi(x, t)) + A(x, x)\nabla^2_{xx}\eta(x, y, t)\big] \\
&\qquad + [\phi(y, t)\nabla_y\eta(x, y, t) + A(y, y)\nabla^2_{yy}\eta(x, y, t) \\
&\qquad + A(x, y)\nabla^2_{xy}\eta(x, y, t) + A(y, x)\nabla^2_{yx}\eta(x, y, t)] = 0.
\end{aligned} \tag{2.11}$$

Note that the first square bracket corresponds to the determining equation for the OPP; indeed, if we assume $\eta(x, y, t)$ does not actually depend on y, all the terms in the second square bracket cancel out (as they should) and we are reduced to the one-particle situation.

Let us now pass on to compute the expression for $(1.11b)$ in the present case. First of all, looking at (2.4) we note that $\sigma^j_k = 0$ for $k > n$. This implies that $(1.11b)$ with given (i, k) reduces to $\sigma^i_p B^p_k = 0$ whenever $k > n$. We have to perform some more computations in the case $k \leqslant n$. This condition will be understood from now on unless specified otherwise.

We use the explicit expressions for σ and ξ in terms of ρ and η (see (2.4) and (2.5)), and write ρ_k for the n vector of components $(\rho^1_k, \dots, \rho^n_k)$, so that, for example, $\rho_k(x)\nabla_x := \rho^j_k(\partial/\partial x^j)$.

We have that $(\sigma^j_k \cdot \partial_j) = \rho_k(x)\nabla_x + \rho_k(y)\nabla_y$, and moreover $(\xi^j \cdot \partial_j)\sigma^i_k = \eta(x, y)\nabla_x \rho_k(x) + \eta(y, x)\nabla_y \rho_k(y)$. Therefore,

$$\{\sigma_k, \xi\}^i = \begin{cases} \rho_k(x)\nabla_x \eta^i(x, y) + \rho_k(y)\nabla_y \eta^i(x, y) - \eta(x, y)\nabla_x \rho^i_k(x) & \text{for} \quad i \leqslant n \\ \rho_k(x)\nabla_x \eta^{i-n}(y, x) + \rho_k(y)\nabla_y \eta^{i-n}(y, x) - \eta(y, x)\nabla_y \rho^{i-n}_k(y) & \text{for} \quad i > n. \end{cases} \tag{2.12}$$

Fixing $k \leqslant n$ and considering the index $i = 1, \dots, 2n$ as a vector one, this can also be rewritten in n-vector notation as

$$\{\sigma_k, \xi\} = \begin{pmatrix} \{\rho_k(x), \eta(x, y)\}_x + \rho_k(y)\nabla_y \eta(x, y) \\ \{\rho_k(y), \eta(y, x)\}_y + \rho_k(x)\nabla_x \eta(y, x) \end{pmatrix}. \tag{2.13}$$

The next two terms in $(1.11b)$ are immediately written in this notation as

$$\tau \partial_t \sigma^i_k = \begin{pmatrix} \tau \partial_t \rho_k(x) \\ \tau \partial_t \rho_k(y) \end{pmatrix} \qquad (\partial_t \tau)\sigma^i_k = \begin{pmatrix} (\partial_t \tau)\rho_k(x) \\ (\partial_t \tau)\rho_k(y) \end{pmatrix}. \tag{2.14}$$

As for the last term, i.e. $\sigma^i_p B^p_k$, we note that in view of the present framework, we should expect that B does not mix the first n components of the $(2n)$-dimensional vector Wiener process $\boldsymbol{v}$ with the other ones, and the transformation of the latter ones can be completely disregarded as they should and will play no role. Thus we should assume

$$B = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.15}$$

Indeed, if we write in full generality

$$B = \begin{pmatrix} M & M_2 \\ M_3 & M_4 \end{pmatrix} \tag{2.16}$$

we obtain

$$\sigma B = \begin{pmatrix} \rho(x)M & \rho(x)M_2 \\ \rho(y)M & \rho(y)M_2 \end{pmatrix}. \tag{2.17}$$

Note that the terms in the last block column correspond to $k > n$ and should therefore be set equal to zero (as discussed above) to satisfy the determining equation, which means $M_2 = 0$; the antisymmetry of B implies then $M_3 = 0$ as well. The $(n \times n)$ matrix M_4 remains undetermined, but it corresponds to an internal transformation of the 'ghost' components $v^{n+1}, \dots, v^{2n}$ of the vector Wiener process, which play no role in the problem and should be disregarded to avoid having spurious symmetries (which are, however, mapped to the identity transformation in terms of the original two-particle problem).

Thus, the term $\sigma^i_p B^p_k$ reads, for $k \leqslant n$ and in the above notation,

$$\sigma^i_p B^p_k = \begin{pmatrix} \rho_j(x) M^j_k \\ \rho_j(y) M^j_k \end{pmatrix}. \tag{2.18}$$

Putting together (2.13), (2.14) and (2.18) we finally obtain (1.11*b*). Note that with (2.15) for B, the equations for $k > n$ are identically satisfied. As for the other ones, note that (for k fixed) those for $i = n+1, \ldots, 2n$ can be obtained from those for $i = 1, \ldots, n$ simply by exchanging x and y, as can be checked from (2.13), (2.14) and (2.18). Thus we can write down only those for $i \leqslant n$, which we do (recalling that $\rho_k \nabla_x := \rho^j_k (\partial/\partial x^j)$, to avoid any confusion) in the notation employed above:

$$\left[\{\rho_k(x), \eta(x,y)\}_x - \tau \partial_t \rho_k(x) - \tfrac{1}{2}\rho_k(x)\partial_t \tau - \rho_j(x) M^j_k\right] + \rho_k(y) \nabla_y \eta(x,y) = 0. \tag{2.19}$$

Note that the terms in square brackets corresponds to the determining equation for the OPP; indeed, if we assume that $\eta(x, y, t)$ does actually not depend on y, the term outside the square brackets cancels out (as it should) and we are reduced to the one-particle situation.

We summarize the result of our computations in the following statement, where we also reintroduce the t dependence for completeness.

Proposition 2. *The determining equations for symmetry generators of the RDS defined by the Ito equation* $\mathrm{d}x^i = \phi^i(x,t)\,\mathrm{d}t + \rho^i_k(x,t)\,\mathrm{d}w^k$ *are*

$$\begin{aligned}
&\left[\{\phi(x,t), \eta(x,y,t)\}_x + \partial_t \eta(x,y,t) - \partial_t(\tau(t)\phi(x,t)) + A(x,x)\nabla^2_{xx}\eta(x,y,t)\right] \\
&\qquad + \left[\phi(y,t)\nabla_y \eta(x,y,t) + A(y,y)\nabla^2_{yy}\eta(x,y,t)\right. \\
&\qquad \left. + A(x,y)\nabla^2_{xy}\eta(x,y,t) + A(y,x)\nabla^2_{yx}\eta(x,y,t)\right] = 0 \\
&\left[\{\rho_k(x,t), \eta(x,y,t)\}_x - \tau(t)\partial_t \rho_k(x,t) - \tfrac{1}{2}\rho_k(x,t)\partial_t \tau(t) - \rho_j(x,t) M^j_k(t)\right] \\
&\qquad + \rho_k(y,t)\nabla_y \eta(x,y,t) = 0.
\end{aligned} \tag{2.20}$$

Remark 6. The terms in square brackets coincide with the left-hand sides of equations (1.11).

It is obvious that, as can also be checked by setting $\nabla_y \eta(x, y, t) = 0$ in the above equations, we have the following:

Corollary. *Any symmetry of the OPP defined by an Ito equation is also a symmetry of the RDS defined by the same Ito equation.*

However, it appears from (2.20) that we can have symmetries of an RDS which are *not* symmetries of the OPP defined by the same Ito equation. In particular, if a solution η of (2.20) depends on the y variables, this cannot be interpreted in terms of the OPP.

The simple examples given below (in section 4) show that indeed an RDS admits in general more symmetries than the corresponding OPP.

2.2. Symmetry of N-particle processes

The whole previous discussion concerned two-particle processes; we should, however, consider N-particle processes, with $N > 1$ an arbitrary integer. The computations in this case are actually completely analogous to those for the two-particle processes; the main effort will be devoted to setting a suitable notation, and once this is done we will briefly go over the computations without reproducing all the details.

We start again by (2.1); we will consider N copies of it, and variables $\boldsymbol{x}_{(\alpha)} \in \mathbb{R}^n$, with $\alpha = 1, \ldots, N$ and $\boldsymbol{x}_{(\alpha)} = (x^1_{(\alpha)}, \ldots, x^n_{(\alpha)})$. All the equations (and results) to be obtained will be covariant under an arbitrary permutation of the $\boldsymbol{x}_{(\alpha)}$ variables (recall the N particles do not interact in any way), and we could therefore just focus on one block, conventionally called the first, of these.

We will consider an (Nn)-dimensional vector $\boldsymbol{z} = (\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(N)})$, a (Nn)-dimensional Wiener process $\boldsymbol{v}$ having as first n components the n-dimensional Wiener process $\boldsymbol{w}$ (the other components will be inessential), and, similarly to what we have done above for $N = 2$, rewrite the system

$$\mathrm{d}x^i_{(\alpha)} = \phi^i(\boldsymbol{x}_{(\alpha)}, t)\,\mathrm{d}t + \rho^i_k(\boldsymbol{x}_{(\alpha)}, t)\,\mathrm{d}w^k \qquad \alpha = 1, \ldots, N \tag{2.21}$$

(note the w^k appearing in the different block of equations are the same) of N Ito equations, each of them n-dimensional, as a single (Nn)-dimensional Ito equation,

$$\mathrm{d}z^i = f^i(\boldsymbol{z}, t)\,\mathrm{d}t + \sigma^i_k(\boldsymbol{z}, t)\,\mathrm{d}v^k. \tag{2.22}$$

The (Nn)-dimensional vector f and matrix σ appearing here are, in n-block notation,

$$f = \begin{pmatrix} \phi(\boldsymbol{x}_{(1)}, t) \\ \vdots \\ \phi(\boldsymbol{x}_{(N)}, t) \end{pmatrix} \qquad \sigma = \begin{pmatrix} \rho(\boldsymbol{x}_{(1)}, t) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \rho(\boldsymbol{x}_{(N)}, t) & 0 & \ldots & 0 \end{pmatrix}. \tag{2.23}$$

In order to simplify the notation, we will write $\phi[\alpha]$ for $\phi(\boldsymbol{x}_{(\alpha)}, t)$, and similarly $\rho[\alpha]$ for $\rho(\boldsymbol{x}_{(\alpha)}, t)$. In this notation we have

$$f = \begin{pmatrix} \phi[1] \\ \vdots \\ \phi[N] \end{pmatrix} \qquad \sigma = \begin{pmatrix} \rho[1] & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \rho[N] & 0 & \ldots & 0 \end{pmatrix}. \tag{2.24}$$

We will also denote by $f^{[\alpha]}$ the αth block component of f, and similarly for other quantities; thus (2.23) and (2.24) above read simply, in this notation, $f^{[\alpha]} = \phi[\alpha]$, $\sigma^{[\alpha]}_{[\beta]} = \rho[\alpha]\delta_{\beta,1}$.

With this same notation, we have that

$$\tfrac{1}{2}\sigma\sigma^T = \begin{pmatrix} A[1,1] & \ldots & A[1,N] \\ \vdots & \ddots & \vdots \\ A[N,1] & \ldots & A[N,N] \end{pmatrix} \tag{2.25}$$

where

$$A[\alpha,\beta] := \tfrac{1}{2}\rho[\alpha]\rho^T[\beta]. \tag{2.26}$$

We will also write the differentiation operators in block notation; ∇_α will be the gradient with respect to the $\boldsymbol{x}_{(\alpha)}$ variables, and

$$A[\alpha,\beta]\nabla^2_{\alpha\beta} := \sum_{i=1}^{n}\sum_{j=1}^{n} \tfrac{1}{2}\big(\rho[\alpha]\rho^T[\beta]\big)^{ij} \frac{\partial^2}{\partial x^i_{(\alpha)}\partial x^j_{(\beta)}}. \tag{2.27}$$

We can now proceed along the same steps as in the $N = 2$ case.

The term $\{f, \xi\}$ is readily evaluated using the fact that $\nabla_\beta\phi[\alpha]$ is zero for $\beta \neq \alpha$: in block notation,

$$\begin{aligned} \{f,\xi\}^{[\alpha]} &= (\phi[\beta]\cdot\nabla_\beta)\xi^{[\alpha]} - (\xi^{[\beta]}\cdot\nabla_\beta)\phi[\alpha] \\ &= \big[(\phi[\alpha]\cdot\nabla_\alpha)\xi^{[\alpha]} - (\xi^{[\alpha]}\cdot\nabla_\alpha)\phi[\alpha]\big] + \sum_{\beta\neq\alpha}(\phi[\beta]\cdot\nabla_\beta)\xi^{[\alpha]}. \end{aligned} \tag{2.28}$$

The evaluation of the terms $\partial_t \xi$ and $\partial_t(\tau f)$ does not present any problem, giving, respectively, $\partial_t \xi^{[\alpha]}$ and $\partial_t(\tau\phi[\alpha])$, and the second derivatives terms is simply $\sum_{\alpha,\beta} A[\alpha,\beta]\nabla^2_{\alpha\beta}\xi^{[\alpha]}$. Summarizing, (1.11*a*), is now, in block notation and with no sum on α,

$$\begin{aligned}\partial_t\xi^{[\alpha]} + \{\phi[\alpha],\xi^{[\alpha]}\}_\alpha + \sum_{\beta\neq\alpha}(\phi[\beta]\cdot\nabla_\beta)\xi^{[\alpha]} - \partial_t(\tau\phi[\alpha]) + \sum_{\beta,\gamma} A[\beta,\gamma]\nabla^2_{\beta\gamma}\xi^{[\alpha]}\\ = \left[\partial_t\xi^{[\alpha]} + \{\phi[\alpha],\xi^{[\alpha]}\}_\alpha - \partial_t(\tau\phi[\alpha]) + A[\alpha,\alpha]\xi^{[\alpha]}\right]\\ + \sum_{\beta\neq\alpha}(\phi[\beta]\cdot\nabla_\beta)\xi^{[\alpha]} + \sum_{(\beta,\gamma)\neq(\alpha,\alpha)} A[\beta,\gamma]\nabla^2_{\beta\gamma}\xi^{[\alpha]} = 0.\end{aligned}\tag{2.29}$$

Note that the square brackets correspond to the equation for the one-particle process.

Let us now consider (1.11*b*). We will use the same block notation.

We will write B in block form as

$$B = \begin{pmatrix} R[1,1] & \dots & R[1,N]\\ \vdots & \ddots & \vdots\\ R[N,1] & \dots & R[N,N]\end{pmatrix}\tag{2.30}$$

note that the $R[\alpha,\alpha]$ are antisymmetric matrices, while in general $R[\alpha,\beta] = R^T[\beta,\alpha]$. For later reference, the submatrix $R[1,1]$ will also be denoted as M. The matrix σB will thus have block components

$$(\sigma B)^{[\alpha]}_{[\beta]} = \rho[\alpha]\, B[1,\beta].\tag{2.31}$$

The term $\{\sigma_k,\xi\}^i \equiv \sigma^j_k\partial_j\xi^i - \xi^j\partial_j\sigma^i_k$ is written, in block notation, as

$$\left(\sigma^{[\gamma]}_{[\beta]}\cdot\nabla_\gamma\right)\xi^{[\alpha]} - \left(\xi^{[\gamma]}\cdot\nabla_\gamma\right)\sigma^{[\alpha]}_{[\beta]}\tag{2.32}$$

by observing that $\sigma^{[\alpha]}_{[\beta]} = \rho[\alpha]\delta_{\beta,1}$, we find that this is trivially zero for $\beta > 1$, and for $\beta = 1$ it is easily written as $(\rho[\gamma]\cdot\nabla_\gamma)\xi^{[\alpha]} - (\xi^{[\gamma]}\cdot\nabla_\gamma)\rho[\alpha]$.

As $\nabla_\gamma\rho[\alpha] = 0$ for $\gamma\neq\alpha$, this can also be rewritten, in the same notation as above and with no sum on α, as

$$\{\rho[\alpha],\xi^{[\alpha]}\}_\alpha + \sum_{\gamma\neq\alpha}(\rho[\gamma]\cdot\nabla_\gamma)\xi^{[\alpha]}.\tag{2.33}$$

The terms $\tau\partial_t\sigma^i_k$ and $(\partial_t\tau)\sigma^i_k$ also vanish for $k > n$ and are easily written for $k\leqslant n$.

Let us summarize our discussion for (1.11*b*). For $k > n$, this reduces, in block notation, to

$$(\sigma B)^{[\alpha]}_{[\beta]} = \rho[\alpha]\, R[1,\beta] = 0.\tag{2.34}$$

If ρ is of rank n, this implies $R[1,\beta] = 0$ for $\beta\neq 1$. Due to the antisymmetry of B, this means that the only non-zero block submatrices are the diagonal ones, $R[\alpha,\alpha]$. Note, however, that for $\alpha\neq 1$ these represent internal transformations of the 'ghost' n-dimensional Wiener processes $(v^{nk+1},\dots,v^{nk+n})$ $(k\geqslant 1)$ which play no role in the problem; as in the $N = 2$ case, these would give spurious symmetries, mapped to the identity in terms of the original N-particle problem, and should be disregarded. Thus the only non-zero relevant submatrix in B is the $R[1,1] = M$ block.

For $k\leqslant n$ equation (1.11*b*) is written, in block notation, as

$$\left[\{\rho[\alpha],\xi^{[\alpha]}\}_\alpha - \tau\partial_t\rho[\alpha] - \tfrac{1}{2}(\partial_t\tau)\rho[\alpha] - \rho[\alpha]R[1,1]\right] + \sum_{\gamma\neq\alpha}(\rho[\gamma]\cdot\nabla_\gamma)\xi^{[\alpha]} = 0.\tag{2.35}$$

Again, the square brackets correspond to the equation for the one-particle process.

We will now summarize our results for the N-particle process defined by an Ito equation in the same way as we did for the two-particle process.

Proposition 3. *The determining equations for symmetry generators of the N-particle process defined by the Ito equation* $\mathrm{d}x^i = \phi^i(\boldsymbol{x}, t)\,\mathrm{d}t + \rho^i_k(\boldsymbol{x}, t)\,\mathrm{d}w^k$ *are*

$$\begin{aligned}&\left[\partial_t \xi^{[\alpha]} + \{\phi[\alpha], \xi^{[\alpha]}\}_{[\alpha]} - \partial_t(\tau\phi[\alpha]) + A[\alpha,\alpha]\nabla^2_{\alpha\alpha}\xi^{[\alpha]}\right] \\ &\qquad + \sum_{\beta\neq\alpha}(\phi[\beta]\cdot\nabla_\beta)\xi^{[\alpha]} + \sum_{(\beta,\gamma)\neq(\alpha,\alpha)} A[\beta,\gamma]\nabla^2_{\beta\gamma}\xi^{[\alpha]} = 0 \\ &\left[\{\rho[\alpha], \xi^{[\alpha]}\}_{[}\alpha] - \tau\partial_t\rho[\alpha] - \tfrac{1}{2}(\partial_t\tau)\rho[\alpha] - \rho[\alpha]M\right] + \sum_{\gamma\neq\alpha}(\rho[\gamma]\cdot\nabla_\gamma)\xi^{[\alpha]} = 0.\end{aligned} \tag{2.36}$$

Remark 7. Note that the terms in square brackets in the two equations (2.36) are nothing else but the terms which have to vanish for the determining equations for symmetries of the OPP defined by the Ito equation (2.1) to be satisfied. For greater clarity we could use the notation $\xi^{[1]} \equiv \eta(\boldsymbol{z}, t)$ and write only the $\alpha = 1$ block component of the equations; as already remarked, the other block components can be recovered by the requirement of covariance of the equations under arbitrary permutations of the block indices. We could also write $\boldsymbol{x}$ for $\boldsymbol{x}_{(1)}$. In this notation, these two terms are written as

$$\begin{aligned}&\partial_t\eta + \{\phi(\boldsymbol{x}, t), \eta\}_{\boldsymbol{x}} - \partial_t[\tau\phi(\boldsymbol{x}, t)] + A(\boldsymbol{x}, \boldsymbol{x})\nabla^2_{\boldsymbol{x}\boldsymbol{x}}\eta \\ &\{\rho(\boldsymbol{x}, t), \eta\}_{\boldsymbol{x}} - \tau\partial_t\rho(\boldsymbol{x}, t) - \tfrac{1}{2}(\partial_t\tau)\rho(\boldsymbol{x}, t) - \rho(\boldsymbol{x}, t)M\end{aligned} \tag{2.37}$$

which coincide with (1.11).

By setting in the previous equations $\eta = \eta(\boldsymbol{x}, t)$, instead of the general case $\eta = \eta(\boldsymbol{z}, t)$, we have the following:

Corollary. *Any symmetry of the OPP defined by an Ito equation is also a symmetry of the N-particle process defined by the same Ito equation.*

Needless to say, it appears from (2.36) that we can have symmetries of an N-particle process which are *not* symmetries of the OPP defined by the same Ito equation, as we already knew from the discussion of the two-particle process.

3. Examples I: W-symmetries

In this and the following sections, we will illustrate the results of previous sections by some simple examples. We start by considering, in this section, examples of W-symmetries for some (OPP defined by) Ito equations; these are just the multidimensional examples considered in [4], so that the symmetries not involving transformations of the $\boldsymbol{w}$ can be read from there.

Example 3.1. Let us consider the two-dimensional Ito equation

$$\begin{aligned}\mathrm{d}x &= y\,\mathrm{d}t \\ \mathrm{d}y &= -k^2 y\,\mathrm{d}t + \sqrt{2k^2}\,\mathrm{d}w(t)\end{aligned} \tag{3.1}$$

this is example 4 of [4]. The vector ξ and the antisymmetric matrix B will be written as

$$\xi = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \qquad B = \begin{pmatrix}0 & b \\ -b & 0\end{pmatrix}. \tag{3.2}$$

If we look at the second set of determining equations and single out that for $i = 2$, $k = 1$ we have immediately that $b = 0$, i.e. in this case we have no new symmetry by allowing transformation of the $\boldsymbol{w}$.

Example 3.2. We now consider the equation, again two-dimensional, given as example 5 in [4]; this is

$$\begin{aligned} \mathrm{d}x &= (a_1/x)\,\mathrm{d}t + \mathrm{d}w_1(t) \\ \mathrm{d}y &= a_2\,\mathrm{d}t + \mathrm{d}w_2(t). \end{aligned} \tag{3.3}$$

We use the same notation (3.2) as above for ξ and B, and pass directly to consider the second set of determining equations, (1.11b). As now $\sigma = I$, these read simply

$$\partial_k \xi^i - (\tau_t/2)\delta^i_k - B^i_k = 0 \tag{3.4}$$

and therefore we have, writing $\tau_t = 2\theta(t)$,

$$\begin{aligned} \alpha_x &= \theta \qquad && (i, k = 1, 1) \\ \alpha_y &= b && (i, k = 1, 2) \\ \beta_x &= -b && (i, k = 2, 1) \\ \beta_y &= \theta && (i, k = 2, 2). \end{aligned} \tag{3.5}$$

These mean that we can write

$$\begin{aligned} \alpha &= \theta x + by + \alpha_0(t) \\ \beta &= -bx + \theta y + \beta_0(t). \end{aligned} \tag{3.6}$$

The first set of determining equations, (1.11a), reads in this case as

$$\begin{aligned} &\alpha_t + \left[\frac{a_1}{x}\alpha_x + a_2\alpha_y + \frac{a_1}{x^2}\alpha\right] - \frac{a_1}{x}\tau_t + \frac{1}{2}(\alpha_{xx} + \alpha_{yy}) = 0 \\ &\beta_t + \left[\frac{a_1}{x}\beta_x + a_2\beta_y\right] - a_2\tau_t + \tfrac{1}{2}(\beta_{xx} + \beta_{yy}) = 0 \end{aligned} \tag{3.7a}$$

which with (3.6) reduce to

$$\begin{aligned} &\frac{1}{2}\tau_{tt}x + \alpha_0' + ba_2 + \frac{a_1by}{x^2} + \frac{a_1\alpha_0}{x^2} = 0 \\ &\frac{1}{2}\tau_{tt}y + \beta_0' - \frac{a_1b}{x} + \frac{a_2}{2}\tau_t - a_2\tau_t = 0. \end{aligned} \tag{3.7b}$$

These imply, as the unknown functions depend on t alone,

$$\tau_{tt} = 0 \qquad \alpha_0' = -ba_2 \qquad \beta_0' = a_2\tau_t/2 \tag{3.8a}$$

$$a_1\alpha_0 = 0 \qquad a_1b = 0 \tag{3.8b}$$

and for $a_1 \neq 0$ we also have $\alpha_0 = 0$, $b = 0$. Thus, unless $a_1 = 0$, we do not have any new symmetry with respect to the case analysed in [4].

In the case $a_1 = 0$ we are left with (3.8a) alone, as (3.8b) are automatically satisfied. Writing $\tau = 2c_1t + c_2$ we obtain immediately $\alpha_0 = -a_2bt + c_3$ and $\beta_0 = c_1a_2t + c_4$, where b and the c_i's are arbitrary real constants.

Thus, for $a_1 = 0$ we have five symmetries; four of these (associated with the arbitrary constants c_i) do not involve the $\boldsymbol{w}$ vector and are given by $X_1 = 2t\partial_t + x\partial_x + (y + a_2t)\partial_y$, $X_2 = \partial_t$, $X_3 = \partial_x$, $X_4 = \partial_y$; the fifth one is a new symmetry with a non-zero B matrix and is given by

$$X_5 = (a_2t - y)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + w_2\frac{\partial}{\partial w_1} - w_1\frac{\partial}{\partial w_2}.$$

Example 3.3. We will now consider the next example in [4], i.e. the two-dimensional Ito system

$$\begin{aligned} \mathrm{d}x &= \cos(t)\,\mathrm{d}w_1 - \sin(t)\,\mathrm{d}w_2 \\ \mathrm{d}y &= \sin(t)\,\mathrm{d}w_1 + \cos(t)\,\mathrm{d}w_2. \end{aligned} \tag{3.9}$$

We will use again the same notation (3.2) for ξ and B.

Note that now $f = 0$, so that the first set of determining equations gives simply

$$\alpha_{xx} + \alpha_{yy} = 0 \qquad \beta_{xx} + \beta_{yy} = 0. \tag{3.10}$$

As for the second set, we obtain

$$\begin{aligned}
&\cos(t)\alpha_x + \sin(t)\alpha_y + \sin(t)\tau - \cos(t)(\tau_t/2) - b\sin(t) = 0 && (i,k=1,1) && (e11)\\
&-\sin(t)\alpha_x + \cos(t)\alpha_y - \cos(t)\tau + \sin(t)(\tau_t/2) - b\cos(t) = 0 && (i,k=1,2) && (e12)\\
&\cos(t)\beta_x + \sin(t)\beta_y + \cos(t)\tau - \sin(t)(\tau_t/2) + b\cos(t) = 0 && (i,k=2,1) && (e21)\\
&-\sin(t)\beta_x + \cos(t)\beta_y - \sin(t)\tau - \cos(t)(\tau_t/2) - b\sin(t) = 0 && (i,k=2,2). && (e22)
\end{aligned} \tag{3.11}$$

We will denote by $\partial_x(e11)$ the equation obtained differentiating $(e11)$ with respect to x, and so on for similar expressions.

By considering $\sin(t)\partial_x(e11) + \cos(t)\partial_y(e11)$, we obtain

$$\sin(t)\cos(t)[\alpha_{xx} + \alpha_{yy}] + \alpha_{xy} = 0 \tag{3.12}$$

recalling (3.10), this means that $\alpha_{xy} = 0$. Similarly, by considering $\sin(t)\partial_x(e21) + \cos(t)\partial_y(e21)$ we obtain $\beta_{xy} = 0$. By considering $\sin(t)\partial_y(e11) + \cos(t)\partial_y(e12)$ we obtain $\alpha_{yy} = 0$, and thus by (3.10) $\alpha_{xx} = 0$ as well. Similarly, $\sin(t)\partial_y(e21) + \cos(t)\partial_y(e22)$ yields $\beta_{xx} = \beta_{yy} = 0$. Thus, α and β are at most linear in x and y.

We use again combinations of the equations given in (3.11):

$$\begin{aligned}
&\sin(t)(e11) + \cos(t)(e12): && \alpha_y + \tau[\sin^2(t) - \cos^2(t)] - b = 0\\
&\sin(t)(e21) + \cos(t)(e22): && \beta_y - \tau_t/2 = 0\\
&\cos(t)(e11) - \sin(t)(e12): && \alpha_x + 2\sin(t)\cos(t)\tau - (\tau_t/2) = 0\\
&\cos(t)(e21) - \sin(t)(e22): && \beta_x + \tau + b = 0.
\end{aligned} \tag{3.13}$$

In this way we obtain

$$\begin{aligned} \alpha &= [\tau_t/2 - \sin(2t)\tau]x + [b + \cos(2t)\tau]y + \alpha_0(t) \\ \beta &= -[b+\tau]x + [\tau_t/2]y + \beta_0(t) \end{aligned} \tag{3.14}$$

one can check, for example, by substituting in (3.11), that this is the most general solution to the determining equations.

By setting $\tau = \alpha_0 = \beta_0 = 0$, we obtain the new (with respect to [4]) symmetry, corresponding to $\alpha = by$ and $\beta = -bx$ with b arbitrary, i.e. given by

$$X = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \left(w^2\frac{\partial}{\partial w^1} - w^1\frac{\partial}{\partial w^2}\right). \tag{3.15}$$

Note that this is nothing other than a simultaneous (and identical) rotation in the (x, y) and in the (w_1, w_2) planes.

Example 3.4. We will now consider n uncoupled equal 'Langevin harmonic oscillators' subject to independent stochastic noises of possibly different strength. This system is written as

$$\mathrm{d}x^i = -x^i\,\mathrm{d}t + \sqrt{2s_i}\,\mathrm{d}w^i \qquad \text{(no sum on } i) \tag{3.16}$$

and is considered, for example, in [5]; obviously $s_i \geqslant 0$, and we assume here that they are strictly greater than zero.

The first set of determining equations (1.11a) reads now as

$$\xi^i_t + \xi^i - x^j \xi^i_j + \tau_t x^i + s_k \xi^i_{kk} = 0 \tag{3.17}$$

while the second set (1.11b) is (no sum on i, k)

$$s_k \xi^i_k = s_i B^i_k + \tfrac{1}{2} s_i \tau_t \delta_{ik}. \tag{3.18}$$

The latter gives at once

$$\xi^i = C^i_j(t) x^j + \alpha^i(t) \tag{3.19}$$

where the matrix C is given by (again no sum on i, k)

$$C^i_k = \frac{1}{2}\tau_t \delta^i_k + \frac{s_i}{s_k} B^i_k. \tag{3.20}$$

Inserting this into (3.17), the terms containing B cancel out and we obtain

$$\tau_{tt} = -2\tau_t \qquad \alpha^i_t = -\alpha^i \tag{3.21}$$

these are known from example 7 of [4] and give symmetries studied there. Setting $\tau = \alpha^i = 0$ to discard these known symmetries, we still have the solution corresponding to B an arbitrary (real, antisymmetric) constant matrix and $\xi^i = C^i{}_k x^k$, where $C^i{}_k = (s_i/s_k) B^i_k$ (no sum on i, k here).

The meaning of this result is obvious: we can act on this system by an arbitrary $SO(n)$ rotation in the $\boldsymbol{w}$ space, while operating a related rotation in the $\boldsymbol{x}$ space.

If we assume $s_1 = s_2 = \cdots = s_n = s$, we have indeed $C = B$ and the rotations in the $\boldsymbol{w}$ and $\boldsymbol{x}$ spaces do just coincide. For general $s_1, \ldots, s_n$, the (s_i/s_k) factors relating the C and B matrices are also easily understood: we could rescale each of the x^i by a factor $\sqrt{2s_i}$, $x^i = \sqrt{2s_i}\, y^i$ (no sum on i), arriving at the manifestly rotationally invariant n-dimensional Ito equation $\mathrm{d}y^i = -y^i\,\mathrm{d}t + \mathrm{d}w^i$.

Example 3.5. Let us consider systems in which the drift $f(x, t)$ is a time-independent linear function of the x, and σ is constant and invertible; i.e.

$$\mathrm{d}x^i = M^i_j x^j\,\mathrm{d}t + \sigma^i_k\,\mathrm{d}w^k \tag{3.22}$$

with M and σ constant matrices.

In this case (1.11b) is simply

$$\sigma^j_k \xi^i_j - (\tau_t/2)\sigma^i_k - \sigma^i_p B^p_k = 0 \tag{3.23}$$

differentiating this in x^m and multiplying from the left-hand side by $(\sigma^T)^{-1}$ we obtain $\xi^i_{jm} = 0$, i.e.

$$\xi^i = L^i_j(t)\, x^j + P^i(t) \tag{3.24}$$

(cf example 8 in [4]). We can thus rewrite (1.11a) as a system of three equations in matrix form,

$$\begin{aligned} &\partial_t L + [L, M] = \tau_t M \\ &\partial_t P = MP \\ &L\sigma = \sigma B + (\tau_t/2)\sigma. \end{aligned} \tag{3.25}$$

The latter one implies $L = \sigma B \sigma^{-1} + (\tau_t/2)I$ and inserting this in the first we obtain, with $\widetilde{M} := \sigma^{-1} M \sigma$,

$$\tau_t \widetilde{M} - \big[\widetilde{M}, B\big] - (\tau_{tt}/2I) = 0. \tag{3.26}$$

For $B = 0$ we obtain the symmetries discussed in [4], while for $\tau = 0$ we obtain new symmetries with B satisfying $[B, \widetilde{M}] = 0$.

Example 3.6. Consider the general case where $\sigma = sI$ ($s \neq 0$ a real number). Equation (1.11b) is now

$$\xi^i_k = B^i_k + (\tau_t/2)\delta^i_k \tag{3.27}$$

and (1.11a) reduces to

$$\partial_t(\xi^i - \tau f^i) + \{f, \xi\} = 0. \tag{3.28}$$

We note that setting $\tau = 0$, we obtain $\xi^i = B^i_k x^k$ and hence $\partial_t \xi^i = 0$; thus, for (3.28) to hold, B has to satisfy $\{Bx, f\} = 0$, i.e.

$$B^i_j f^j - B^j_k x^k \partial_j f^i = 0. \tag{3.29}$$

Example 3.7. As a final and again very simple example, we consider the n-dimensional Ito equation

$$\mathrm{d}x^i = -(1 - \lambda ||x||^2) x^i \,\mathrm{d}t + \mathrm{d}w^i \tag{3.30}$$

with λ a non-zero real constant. This was considered as example 10 in [4], where it was shown that the only symmetry (not allowing an action on $\boldsymbol{w}$) was given by time translations.

In the present case, $\sigma = I$, and thus the discussion of the previous example applies. We check that the system is rotationally invariant, i.e. we consider $\tau = 0$. With $\xi^i = B^i_k x^k$, we have in this case

$$\{\xi, f\}^i = \{Bx, (1 - \lambda ||x||^2) x\} = -2\lambda x^i \big(x^j B^j_k x^k\big) = 0 \tag{3.31}$$

the last equality following from the antisymmetry of B. That is, we conclude that (3.30) is indeed symmetric under simultaneous identical rotations of the $\boldsymbol{x}$ and $\boldsymbol{w}$ vectors.

4. Examples II: symmetries of RDS

We will now illustrate the results of the previous section 2, i.e. symmetries of RDS, by some simple examples; these are again equations already considered in [4], so that the OPP symmetries are given there.

As the main interest here is in determining concrete examples of symmetries for an RDS defined by a given Ito equation which are not symmetries of the OPP defined by the same equation, and not in determining the most general symmetry of the given RDS, we will use the ansatz

$$\tau(t) = 0 \qquad \text{and} \qquad \partial_t \eta(x, y, t) = 0 \tag{4.1}$$

to simplify the (severely underdetermined) determining equations (2.20). For ease of notation, we will write simply η for $\eta(x, y, t)$.

Example 4.1. As a first and very simple example, consider the trivial one-dimensional Ito equation

$$\mathrm{d}x = \mathrm{d}w \tag{4.2}$$

now $\phi(x,t) \equiv 0$ and $\rho(x,t) = 1$. With these, and recalling that $B = 0$ for one-dimensional problems, equation (2.20) reduces to

$$\begin{aligned} &\eta_t + \tfrac{1}{2}\left[\eta_{xx} + \eta_{yy} + 2\eta_{xy}\right] = 0 \\ &\eta_x + \eta_y - \tau_t/2 = 0. \end{aligned} \tag{4.3}$$

We can now note immediately that $\eta(x,y,t) = \alpha(x-y)$, $\tau = 0$, is a smooth solution to (4.3) and thus a symmetry for the two-particle process defined by (4.2), for any smooth function α. Obviously, such solutions cannot be interpreted in terms of OPP, i.e., in particular, they are not symmetries of the OPP defined by (4.2).

Example 4.2. We will now consider the Ito equation

$$\mathrm{d}x = \mathrm{d}t + x\,\mathrm{d}w(t) \tag{4.4}$$

i.e. the case $\phi(x,t) = 1$, $\rho(x,t) = x$. The determining equations (2.20) are now

$$\begin{aligned} &\eta_x + \eta_y + \eta_t - \tau_t + \tfrac{1}{2}\left[x^2\eta_{xx} + y^2\eta_{xy} + 2xy\eta_{xy}\right] = 0 \\ &x\eta_x + y\eta_y = \eta + \tfrac{1}{2}x\tau_t \end{aligned} \tag{4.5}$$

with the ansatz $\tau = 0$ the second of these yields immediately $\eta = a(t)x + b(t)y$, and inserting this into the first of (4.5) we obtain $\eta = c(x-y)$ with c an arbitrary constant.

Again, this depends on y and thus cannot be seen as a symmetry (nor interpreted in terms) of the OPP defined by (4.4).

Example 4.3. We next consider the Ito equation

$$\mathrm{d}x = x\,\mathrm{d}t + \mathrm{d}w(t) \tag{4.6}$$

the determining equations (2.20) for this are

$$\begin{aligned} &x\eta_x + y\eta_y + \eta_t - \eta - x\tau_t + \tfrac{1}{2}\left[\eta_{xx} + \eta_{yy} + 2\eta_{xy}\right] = 0 \\ &\eta_x + \eta_y = \tfrac{1}{2}\tau_t. \end{aligned} \tag{4.7}$$

With the ansatz (4.1) we find from the second of these $\eta = \alpha(x-y)$ with α an arbitrary function. Inserting this in the first of (4.7) we obtain again $\eta = c(x-y)$ with c an arbitrary constant; and again, this depends on y and thus cannot be seen as a symmetry (nor interpreted in terms) of the OPP defined by (4.6).

Example 4.4. As an example based on a two-dimensional Ito equation, we consider the system

$$\begin{aligned} &\mathrm{d}x_1 = x_2\,\mathrm{d}t \\ &\mathrm{d}x_2 = -k^2x_2 + \sqrt{2k^2}\,\mathrm{d}w(t) \end{aligned} \tag{4.8}$$

i.e. the system identified by

$$\phi = \begin{pmatrix} x_2 \\ -k^2x_2 \end{pmatrix} \qquad \rho = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2k^2} \end{pmatrix}. \tag{4.9}$$

We will write η as a 2-vector with components (η^1, η^2).

Let us first look at the second set of equations (2.20): that for $(i, k) = (1, 1)$ is identically satisfied; that for $(i, k) = (2, 1)$ yields $B_{21} = 0$, i.e. $B = 0$. Those for $(i, k) = (1, 2)$ and $(i, k) = (2, 2)$ give, respectively,

$$\begin{aligned} &\frac{\partial \eta^1}{\partial x_2} + \frac{\partial \eta^1}{\partial y_2} = 0 \\ &\frac{\partial \eta^2}{\partial x_2} + \frac{\partial \eta^2}{\partial y_2} = \frac{1}{2}\tau_t. \end{aligned} \tag{4.10}$$

With the ansatz (4.1), and writing $z := (x_2 - y_2)$ these yield immediately

$$\eta^1 = \eta^1(x_1, y_1, z) \qquad \eta^2 = \eta^2(x_1, y_1, z). \tag{4.11}$$

The first set of equations (2.20) reads, for our Ito equation (4.8),

$$\begin{aligned} &x_2\frac{\partial \eta^1}{\partial x_1} - k^2 x_2 \frac{\partial \eta^1}{\partial x_2} - \eta^2 + y_2 \frac{\partial \eta^1}{\partial y_1} - k^2 y_2 \frac{\partial \eta^1}{\partial y_2} + \frac{\partial \eta^1}{\partial t} - x_2\tau_t \\ &\qquad + k^2\left[\frac{\partial^2 \eta^1}{\partial x_2^2} + \frac{\partial^2 \eta^1}{\partial y_2^2} + 2\frac{\partial^2 \eta^1}{\partial x_2 \partial y_2}\right] = 0 \\ &x_2\frac{\partial \eta^2}{\partial x_1} - k^2 x_2 \frac{\partial \eta^2}{\partial x_2} + k^2\eta^2 + y_2 \frac{\partial \eta^2}{\partial y_1} - k^2 y_2 \frac{\partial \eta^2}{\partial y_2} + \frac{\partial \eta^2}{\partial t} + k^2 x_2\tau_t \\ &\qquad + k^2\left[\frac{\partial^2 \eta^2}{\partial x_2} + \frac{\partial^2 \eta^2}{\partial y_2} + 2\frac{\partial^2 \eta^2}{\partial x_2 \partial y_2}\right] = 0. \end{aligned} \tag{4.12}$$

Note that with (4.11) the terms in square brackets vanish automatically, and we are reduced to

$$\begin{aligned} &x_2\frac{\partial \eta^1}{\partial x_1} + y_2\frac{\partial \eta^1}{\partial y_1} - k^2 z\frac{\partial \eta^1}{\partial z} = \eta^2 \\ &x_2\frac{\partial \eta^2}{\partial x_1} + y_2\frac{\partial \eta^2}{\partial y_1} - k^2 z\frac{\partial \eta^2}{\partial z} = -k^2\eta^2. \end{aligned} \tag{4.13}$$

These admit an infinity of solutions in the form

$$\eta = h(\zeta)\cdot\begin{pmatrix} x_1 - y_1 \\ 1 \end{pmatrix} \tag{4.14}$$

where h is an arbitrary smooth function of

$$\zeta := k^2(x_1 - y_1) + (x_2 - y_2). \tag{4.15}$$

Example 4.5. We will finally consider a simple but relevant class of n-dimensional Ito equations, i.e. those with a linear drift ϕ and a constant diagonal noise matrix ρ:

$$\mathrm{d}x^i = M^i{}_j x^j\,\mathrm{d}t + \sqrt{2s^2}\,\mathrm{d}w^i(t). \tag{4.16}$$

Here M is a constant $(n \times n)$ matrix and s is a real constant.

Equations (2.20) read in this case, and assuming directly (4.1),

$$\begin{aligned} &M^j{}_k x^k \frac{\partial \eta^i}{\partial x^j} - M^i{}_j\eta^j + M^j{}_k y^k\frac{\partial \eta^i}{\partial y^j} + s\left[\frac{\partial^2 \eta^i}{\partial x^j \partial x^j} + \frac{\partial^2 \eta^i}{\partial y^j \partial y^j} + 2\frac{\partial^2 \eta^i}{\partial x^j \partial y^j}\right] = 0 \\ &\frac{\partial \eta^i}{\partial x^k} + \frac{\partial \eta^i}{\partial y^k} = B^i{}_k \end{aligned} \tag{4.17}$$

from the second of these we have, with $z^i := (x^i - y^i)$,

$$\eta^i = \alpha^i(z^1, \dots, z^n) + B^i{}_k x^k. \tag{4.18}$$

The first of (4.17) is then

$$M^i{}_k z^k \frac{\partial \alpha^i}{\partial z^j} + B^i{}_j M^j{}_k x^k - M^i{}_j \alpha^j - M^i{}_j B^j{}_k x^k. \tag{4.19}$$

As both B and M cannot depend on x or z variables, this splits into two equations,

$$\begin{aligned} &B^i{}_j M^j{}_k = M^i{}_j B^j{}_k \\ &M^j{}_k z^k \left(\frac{\partial \alpha^i}{\partial z^j}\right) = M^i{}_j \alpha^j. \end{aligned} \tag{4.20}$$

In particular, we obtain solutions in the form (4.18) with $\alpha^i(\boldsymbol{z}) = L^i{}_k z^k$ for any pair of matrices B, L which both commute with M,

$$[B, M] = 0 = [L, M]. \tag{4.21}$$

Acknowledgments

I would like to thank Niurka Rodríguez Quintero, with whom I started the study of symmetries of stochastic equations [4]. Our first discussions, and thus eventually this work, were triggered by invitations of L A Ibort in the Universidad Carlos III de Madrid, and by A Sanchez to present my—then very vague—ideas on the subject in his seminar at GISC. I would like to thank them again, as well as A Degasperis for his never failing hospitality at 'La Sapienza'. Last but not least, thanks should go to the anonymous referee of [4] who urged us to study the symmetry of random dynamical systems and not just of one-point processes.

Appendix. Discrete symmetries of Ito equations

In this appendix we briefly consider, for the sake of completeness, *discrete* symmetries of stochastic differential equations. Similarly to what happens in the deterministic case, the resulting determining equations are, in general, too difficult to be attacked except for very simple classes of symmetries (e.g. reflections, or, however, linear ones). On the other hand, they can be used (a) to check whether a given discrete transformation is a symmetry of a given stochastic differential equation and (b) to determine the stochastic differential equations which admit a given discrete transformation as a symmetry. See [3] for a discussion of the deterministic case.

We consider again an Ito equation of the form

$$\mathrm{d}x^i = f^i(\boldsymbol{x}, t)\,\mathrm{d}t + \sigma^i{}_k(\boldsymbol{x}, t)\,\mathrm{d}w^k(t) \tag{A.1}$$

with $i, k = 1, \dots, n$. We will not consider here transformations acting on the t coordinate, and will thus limit ourselves to considering the change of coordinates in the $\boldsymbol{x}$ and $\boldsymbol{w}$ spaces given by

$$y^i = \phi^i(x, t) \qquad w^k(t) = R^k{}_p(t) z^p(t). \tag{A.2}$$

As discussed in section 1, R will be assumed to be a constant orthogonal matrix, $R \in O(n)$.

With the Ito formula, and denoting as usual $A = \frac{1}{2}\sigma\sigma^T$, we have at once

$$\mathrm{d}y^i = \left[\frac{\partial \phi^i}{\partial x^j} f^j + A^{jk} \frac{\partial^2 \phi^i}{\partial x^j \partial x^k} + \frac{\partial \phi^i}{\partial t}\right] \mathrm{d}t + \left[\frac{\partial \phi^i}{\partial x^j} \sigma^j{}_p R^p{}_k\right] \mathrm{d}z^k. \tag{A.3}$$

Note that here $\phi = \phi(x, t)$, $\sigma = \sigma(x, t)$ and R is constant.

Thus, by requiring that this is just (A.1) again, written in terms of $(\boldsymbol{y}, \boldsymbol{z})$ instead than of $(\boldsymbol{x}, \boldsymbol{w})$, we obtain the *determining equations for discrete symmetries of an Ito equation*:

$$\begin{aligned} \frac{\partial\phi^i(x,t)}{\partial x^j}\, f^j(x,t) + A^{jk}(x,t)\, \frac{\partial^2\phi^i(x,t)}{\partial x^j \partial x^k} + \frac{\partial\phi^i(x,t)}{\partial t} &= f^i(\phi(x,t),t) \\ \frac{\partial\phi^i(x,t)}{\partial x^j}\, \sigma^j{}_p(x,t)\, R^p{}_k = \sigma^i{}_k(\phi(x,t),t). & \end{aligned} \tag{A.4}$$

Remark. We stress that we have actually considered symmetries of the one-particle process defined by the Ito equation (A.1); the extension to N-particle processes would go along the lines of the discussion in section 2.

Let us now briefly consider some very simple examples regarding discrete symmetries of Ito equations.

Example A.1. The simplest case of discrete transformation is provided by $\phi^i(x, t) = -x^i$, $R = \pm I$. In the case $R = I$, equation (A.4) reduce to

$$f^i(x,t) = -f^i(-x,t) \qquad \sigma(x,t) = -\sigma(-x,t). \tag{A.5}$$

In the case $\phi^i(x, t) = -x^i$, $R = -I$, we obtain instead

$$f^i(x,t) = -f^i(-x,t) \qquad \sigma(x,t) = \sigma(-x,t). \tag{A.6}$$

Example A.2. We can consider the case of n independent 'Langevin oscillators', see (3.16),

$$\mathrm{d}x^i = -a_{(i)}x^i\,\mathrm{d}t + s_{(i)}\,\mathrm{d}w^i \qquad \text{(no sum on } i) \tag{A.7}$$

with $a_{(i)}, s_{(i)}$ positive constants. It is immediately obvious to check that (A.6) is satisfied. More in general, equations (A.4) now read (no sum over i, sum over j when repeated)

$$\begin{aligned} a_{(j)}\frac{\partial\phi^i}{\partial x_j} - \frac{s_{(j)}^2}{2}\frac{\partial^2\phi^i}{\partial x^j\partial x^j} &= a_{(i)}\phi^i \\ s_{(j)}\frac{\partial\phi^i}{\partial x_j}R^j{}_k &= s_{(i)}\delta^i{}_k. \end{aligned} \tag{A.8}$$

Example A.3. Let us consider the case of *linear* symmetries $\phi^i(x, t) = L^i{}_p(t)x^p$: now (A.4) reduce to

$$\begin{aligned} L^i{}_j(t)\, f^j(x,t) &= f^i(Lx,t) \\ L^i{}_j(t)\,\sigma^j{}_p(x,t)\, R^p{}_k &= \sigma^i{}_k(Lx,t). \end{aligned} \tag{A.9}$$

If we further assume f to also be linear in x, $f^i(x, t) = M^i{}_j(t)x^j$, these read in obvious shorthand notation

$$[L, M] = 0 \qquad L\sigma(x)R = \sigma(Lx) \tag{A.10}$$

when σ does not depend on x and is invertible, we obtain a linear symmetry from any L which commutes with M, with $R = \sigma^{-1}L^{-1}\sigma$.

Example A.4. For equations with $\sigma^i{}_k(x, t) = s_0\delta^i{}_k$ (with s a real constant), the second of (A.4) always reduces to $(\partial\phi^i/\partial x^j) = (R^{-1})^i{}_j$. However, this implies ϕ must be a linear function of the x, $\phi^i(x, t) = L^i{}_j(t)x^j$ with L an orthogonal matrix ($L = R^{-1}$). Thus in this case we are always reduced to the simple equation $L^i{}_j f^j(x, t) = f^i(Lx, t)$, $L \in O(n)$.

References

[1] Arnold L 1998 *Random Dynamical Systems* (Berlin: Springer)

[2] Arnold L and Imkeller P 1998 Normal forms for stochastic differential equations *Prob. Theor. Rel. Fields* **110** 559–88

[3] Gaeta G and Rodríguez M A 1996 Discrete symmetries of differential equations *J. Phys. A: Math. Gen.* **29** 859–80

[4] Gaeta G and Rodríguez Quintero N 1999 Lie-point symmetries and stochastic differential equations *J. Phys. A: Math. Gen.* **32** 8485–505

[5] van Kampen N G 1992 *Stochastic Processes in Physics and Chemistry* 2nd edn (Amsterdam: North-Holland)

[6] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)